

Characterizing Graphs of Zonohedra

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Abstract

A classic theorem by Steinitz states that a graph G is realizable by a convex polyhedron if and only if G is 3-connected planar. Zonohedra are an important subclass of convex polyhedra having the property that the faces of a zonohedron are parallelograms and are in parallel pairs. In this paper we give characterization of graphs of zonohedra. We also give a linear time algorithm to recognize such a graph. In our quest for finding the algorithm, we prove that in a zonohedron P both the number of zones and the number of faces in each zone is $O(\sqrt{n})$, where n is the number of vertices of P .

Key words: Convex polyhedra, zonohedra, Steinitz's theorem, planar graph
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1 Introduction

Polyhedra are fundamental geometric structures in 3D. Polyhedra have fascinated mankind since prehistory. They were first studied formally by the ancient Greeks and continue to fascinate mathematicians, geometers, architects, and computer scientists [9]. Polyhedra are also studied in the field of art, ornament, nature, cartography, and even in philosophy and literature [3].

Usually polyhedra are categorized based on certain mathematical and geometric properties. For example, in platonic solids, which are the most primitive convex polyhedra, vertices are incident to the same number of identical regular polygons, in Archimedean solids the vertices are allowed to have more than

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one type of regular polygons but the sequence of the polygons around each vertex are the same, etc. Among other types of polyhedra, Johnson solids, prisms and antiprisms, zonohedra, Kepler-pointsod polyhedra, symmetrohedra are few to mention. (See [10], the books [2,3], and the webpages [8,15] for more on different classes of polyhedra.)

Polyhedra, in particular convex, provide a strong link between computational geometry and graph theory, and a major credit for establishing this link goes to Steinitz. In 1922, in a remarkable theorem Steinitz stated that a graph is the naturally induced graph of a convex polyhedron if and only if it is 3-connected planar [7,12]. Till today, Steinitz's theorem attracts the scientists and mathematicians to work on it. For example, there exist several proofs of Steinitz's theorem [4,17].

One of the simplest subclasses of convex polyhedra are *generalized zonohedra*, where every face has a parallel face and the edges in each face are in parallel pairs [13] (see Figure 1(a)). This definition of zonohedra is equivalent to the definition given originally by the Russian crystallographer Fedorov [2,13]. Later Coxeter [2] considered two other definitions of zonohedra to mean more special cases: (i) all faces are parallelograms and (ii) all faces are rhombi. Coxeter called these two types as *zonohedra* and *equilateral zonohedra* respectively. The polyhedra of Figures 1(b) and 1(c) are two examples of these two types respectively. (For history and more information on zonohedra, see [13] and the web pages [5,6,8].) In this paper, by *zonohedra* we mean zonohedra defined by Coxeter.

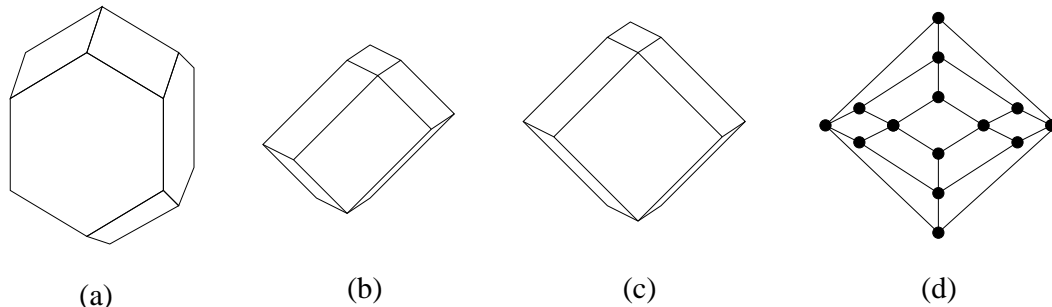


Fig. 1. (a) A generalized zonohedron, (b) a zonohedron, (c) an equilateral zonohedron, and (d) the graph of the zonohedra of (b) and (c).

As mentioned earlier, so far convex polyhedra have been classified in many different classes based on geometric properties. But to our knowledge they have not been classified based on their graphs. Motivated by Steinitz's theorem, in this paper we characterize the graphs of zonohedra (Section 3, 4). Graphs of zonohedra are also called the *zonohedral graphs*. See Figure 1(d). We also give a linear time algorithm for recognizing a zonohedral graph (Section 5). As accompanying results, we show that in a zonohedron P both the number of "zones" and the number of faces in each zone is $O(\sqrt{n})$, where n is the number

of vertices of P (Section 4).

2 Preliminaries

A *graph* $G = (V, E)$ consists of a set of vertices $V(G)$ and a set of edges $E(G)$ where an edge $(u, v) \in E(G)$ connects two vertices $u, v \in V(G)$. A *path* in a graph G is an alternating sequence of distinct vertices and edges where each edge is incident to two vertices immediately preceding and following it. A *cycle* is a closed path in G . The *length* of a path (cycle) is the number of its edges.

A graph G is *connected* if for two distinct vertices u and v there is a path between u and v in G . A (connected) *component* of G is a maximal connected subgraph of G . The *connectivity* $\kappa(G)$ of G is the minimum number of vertices whose removal makes G disconnected or a single vertex. We say that G is *k-connected* if $\kappa(G) \geq k$. Alternatively, G is *k-connected* if for any two vertices of G there are at least k disjoint paths.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at their common extremity. A *plane graph* is a planar graph with a fixed embedding in the plane. A planar graph divides the plane into connected regions called *faces*.

A *convex polyhedron* P is the bounded intersection of a finite number of half-spaces. The bounded planar surfaces of P are called *faces*. The faces meet along line segments, called *edges*, and the edges meet at endpoints, called *vertices*. By Euler's theorem every convex polyhedron with n vertices has $\Theta(n)$ edges and $\Theta(n)$ faces [3]. We call the number of edges belonging to a face (of a polyhedron or a plane graph) as its *face length*.

Let s be the unit sphere centered at the origin. On s the *normal point* of a face f of P is the intersection point of s and the outward normal vector of f drawn from the origin. In the *Gauss map* of P on s each face is represented by its normal point, each edge e is represented by the geodesic arc between the normal points of the two faces adjacent to e , and each vertex v is represented by the spherical convex polygon formed by the arcs of the edges incident to v .

In the *graph of a convex polyhedron* P , there is exactly one vertex for each vertex of P and two vertices are connected by an edge if and only if the corresponding vertices in P form an edge.

Quite frequently we will use the same symbols for terms such as edge, face, zone, etc. that are defined for both a polyhedron and its graph.

3 Characterization

Faces of a zonohedron P are grouped into different “cycles” of faces called the *zones* of P . All faces of a zone z , which are called the *zone faces* of z , are parallel to a direction called the *zone axis* of z . For example, a cube has three zones with three zone axes perpendicular to each other. Every zone face f of z has exactly two edges that are parallel to the zone axis of z , and the set of all such edges of the zone faces of z are called the *zone edges* of z .

In the Gauss map of P a zone z of P is represented as a great circle g of s . Why? The normals of the zone faces of z are perpendicular to the zone axis of z . So the corresponding normal points lie on a great circle whose plane is perpendicular to the zone axis, and g is exactly that great circle. Observe that g is the concatenation of small geodesic arcs corresponding to the zone edges of z .

Lemma 3.1 *A zone z of P has even number of faces.*

Proof. In P any face f has a parallel face f' . Now consider the Gauss map of P where g is the great circle corresponding to the zone z . Since on s the normal points of f and f' are antipodal, if g contains the normal point of f then it also contains the normal point of f' . Equivalently, if f is in z , then f' too is in z , and thus the number of faces in z is even. \square

Lemma 3.2 *Any two zones z_1 and z_2 of P intersect into two parallel faces.*

Proof. Consider the Gauss map of P . Let g_1 and g_2 be the great circles corresponding to the zones z_1 and z_2 respectively. g_1 and g_2 intersect into two antipodal points p_1 and p_2 . Since a pair of antipodal points corresponds to a pair of parallel faces, z_1 and z_2 intersect into nothing but the pair of parallel faces corresponding to p_1 and p_2 . \square

Lemma 3.3 *A face of P belongs to exactly two zones of P .*

Proof. A face of P has two pairs of parallel edges which are zone edges of two different zones. \square

Lemma 3.4 *Let f and f' be two parallel faces of a zone z . Then f and f' divides z into two non-empty equal chains of faces (excluding f and f') where each face in one chain has its parallel pair in the other chain.*

Proof. Consider the Gauss map of P . Let g be the great circle corresponding to the zone z . Since f and f' are parallel, their normal points are two antipodal points p and p' , respectively, in g . p and p' divides g into two half circles h_1 and h_2 . If h_1 contains a normal point $p_1 \notin \{p, p'\}$ of a face f_1 of z , then the

antipodal point of p_1 , which is the normal point of the parallel face of f_1 , is in h_2 . \square

Let us now reflect the above (geometric) properties of P to its graph. Let G be a 3-connected planar graph with even faces. According to Tutte [14,16], every 3-connected planar graph has a unique planar embedding (and from now on by G we mean it with its unique planar embedding.) We override the definition of a zone for G . The pair of alternating edges of a (quadrilateral) face f (of P or G) are called *opposite* to each other in f . A *zone* z in G is a cycle of *zone faces* where for any three consecutive zone faces f_i, f_{i+1}, f_{i+2} the common edge of f_i and f_{i+1} and the common edge of f_{i+1} and f_{i+2} are opposite in f_{i+1} , and these common edges of the zone faces of z are called the *zone edges* of z .

As implied from the above properties of P , the necessary conditions for G to be a graph of a zonohedron are: G is 3-connected planar, its face lengths are four, every face is a zone face of exactly two zones, and any pair of zones intersect into two faces and divide each other into two non-empty equal chains of faces. Our main result consists in providing a characterization by showing that the above conditions are also sufficient.

Theorem 3.5 *A graph G is a graph of a zonohedron iff G is 3-connected planar, faces of G are quadrilaterals, every face is a zone face of exactly two zones, and each pair of zones in G intersect into two faces and divide each other into two non-empty equal chains of faces (excluding the two common faces).*

Note that the property of a 3-connected planar graph having a unique planar embedding [14] is used only for identifying faces of G . It is not explicitly used in the characterization. This is because neither the number of zones nor the number of faces in a zone depend on that unique embedding.

4 Proof of the sufficiency

Our proof is constructive. The idea of our construction is to delete zones one after another from the given graph G until we reach the graph of a cube. The graph of a cube is the smallest graph satisfying the sufficient condition. Then from a cube we construct a zonohedron by adding zones in reverse order one after another.

The pair of faces in which two zones of G intersect is called a *face pair*. We define the *length* of a zone of G as the number of faces in it. Since two zones divide each other into two non-empty equal chains of faces, the length of a

zone is even and is at least four.

4.1 Deleting zones from G

We first define the deletion of a zone z . By *contraction* of a zone edge $e = (u, v)$ we mean the replacement of u and v with a single vertex whose incident edges are the edges (other than e) that were incident to u or v . Let f be a zone face of z and let e_1 and e_2 be two edges of f that are zone edges of z . We define the *contraction* of f as follows. We contract e_1 and e_2 into two vertices w_1 and w_2 respectively. w_1 and w_2 now have two edges between them. We replace these two edges by a single one (and keep the other edges incident to w_1 and w_2 unchanged.) We define the *deletion* of a zone z as the contraction of all zone faces of z . See Figure 2.

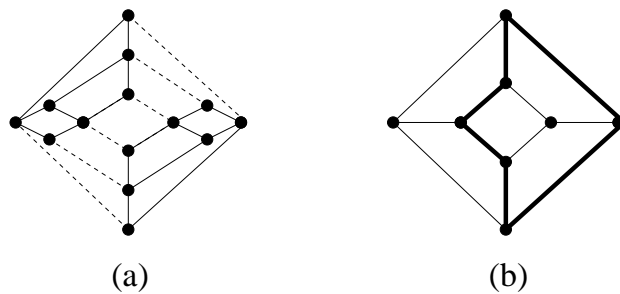


Fig. 2. (a) Illustration of deletion of a zone from G , (b) the graph G' after deletion of the zone. The heavily drawn edges show the cycle resulting from the deleted zone.

Since a face f belongs to exactly two zones, z does not self intersect. It implies that each contracted face of z results into an edge after the deletion. So as a whole, deleting z results in a cycle whose length is same as that of z , and we call this cycle as the *zone cycle* corresponding to z .

We identify all the zones of G as follows. For each face of G by traversing the edges in circular order we find the pair of opposite edges. Starting from a face f , we identify the two zones that have f in common. For each of them we group their adjacent faces one after another based on opposite edges and check any intersection within a zone before we come back to f . Similarly, we approach other face pairs having opposite edges that have not been encountered yet. Since every two zones intersect exactly two faces the total number of times a face is traversed is at most half the number of its edges.

Once we identify the zones of G we delete them one by one until we reach the graph of a cube, for which the following observation is obvious.

Lemma 4.1 G is the graph of a cube iff G has three zones of length four.

Due to the above lemma we only delete the zones of size six or more to reach

the graph of a cube. The following lemma will prove that we can successfully do that.

Lemma 4.2 *Let G be a graph of a zonohedron with more than three zones. Let G' be the graph after deleting a zone z from G . Then G' satisfies the conditions of Theorem 3.5.*

Proof.

To prove that G' is 3-connected, we first show that the new vertex w in G' obtained by contracting an edge $e = (u_1, u_2)$ of z has degree three or more. Suppose for a contradiction that $d(w) < 3$. Since $d(u_1) \geq 3$, $d(u_2) \geq 3$ then $d(w) \not\leq 2$. If $d(w) = 2$ then $d(u_1) = d(u_2) = 3$. Suppose u_1 and u_2 belong to the faces f_1 and f_2 other than the faces f_z and f'_z of z in G . If f_1 (similarly f_2) belongs to zones z_1 and z_2 , then f_2 (f_1) also belongs to z_1 and z_2 as illustrated in Figure 3. According to Theorem 3.5, z_1 and z_2 divide each other into two non-empty equal chains of faces, each having exactly one face. Since each pair of zones intersect each other into two faces the number of zones in G cannot be greater than three, a contradiction to the assumption that G contains more than three zones.

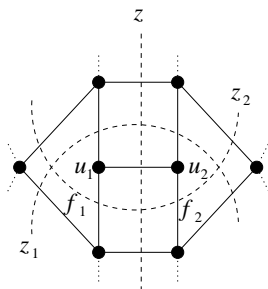


Fig. 3. A zonohedral graph G having $d(u_1) = d(u_2) = 3$ (dashed lines represent the zones).

We now show that, there exist at least three disjoint paths between any two vertices in G' . In G any two vertices u and v had at least three disjoint paths and any contracted edge of z can be in at most one of these paths. Now in G' , if u and v are contracted together, then we are done. If one of them, say u , was contracted with its neighbor to a new vertex w (similarly if none of u and v was contracted), then w (similarly u) maintains those three disjoint paths to v , possibly with smaller path lengths.

By deleting z we have neither modified the faces of G nor introduced new faces. Hence if all the faces of G belongs to exactly two zones then every face of G' belongs to exactly two zones.

Next we show that any two zones in G' divide each other into two non-empty equal chains of faces. Let z_1 and z_2 be two zones other than z in G . Let (f, f')

be the face pair of G at which z_1 and z_2 intersect each other. Let the two equal chains of faces of z_1 between f and f' be l_1 and l_2 . We will show that in G' , l_1 and l_2 have equal length of at least two. A similar argument holds for z_2 and allows to complete the proof.

Consider the intersection of z and z_1 . Let (f_1, f_2) be the face pair of G at which z and z_1 intersect. Since (f_1, f_2) divides z_1 into two other equal chains of faces, f_1 is in l_1 (similarly in l_2) if and only if f_2 is in l_2 (similarly in l_1). W.l.o.g. assume that f_1 is in l_1 . After deleting z , z_1 loses exactly two faces: f_1 from l_1 and f_2 from l_2 . So in G' , l_1 and l_2 are of equal length. Moreover, by Lemma 4.1, in G , z_1 has length at least six. So in G' , l_1 and l_2 have length at least two. \square

4.2 Adding 3D zones

Let the current zonohedron be P' and its graph be G' . Let G be the graph from which G' was obtained by deleting a zone. We will add to P' a zone z corresponding to the deleted zone of G as follows.

Let P be the resulting polyhedron after adding z to P' . Let c be the cycle (of edges) in P' that corresponds to the zone cycle of the deleted zone of G . By Lemmas 4.2 and 3.4, c divides each zone of P' into two equal chains of faces where faces in one chain have parallel pairs in the other chain. As a whole, c divides the set of faces of P' into two subsets P'_1 and P'_2 where faces in P'_1 (P'_2) have parallel pairs in P'_2 (P'_1).

To get P we expand each edge of c to a rhombus in a common direction d . Clearly the graph of P is G . See Figure 4. What remain to be proven are: (i) the faces of z are in parallel pairs, and (ii) there exists a d such that P is convex. We prove them in the following two lemmas respectively.

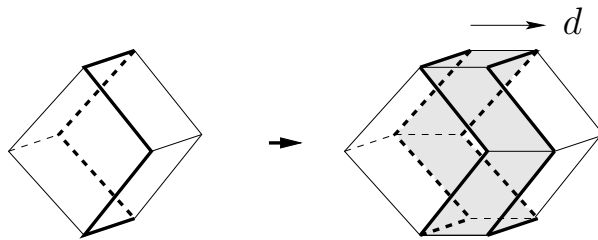


Fig. 4. Adding a pseudo prism to a cube. Heavily drawn lines show c and shaded faces are the newly added zone.

Lemma 4.3 *The faces of z are in parallel pairs.*

Proof. It suffices to show that edges of c are in parallel pairs. Consider an edge e of c . Let z be the zone of which e is a zone edge. By Lemma 4.2, z

crosses c twice. Let e' be the other edge of c at which z crosses c . Since e and e' belong to the same zone z , they are parallel (to the zone axis of z). \square

Lemma 4.4 *There exists d such that P is convex. Moreover, d can be found in $O(h \log h)$ time where h is the number of faces of z .*

Proof. To prove that P is convex it suffices to prove that there exists d such that no face of P' is parallel to d and viewing P' orthogonally from d keeps c as the boundary of the projection (and thus makes all the faces in one side of c visible and the faces in other side invisible). We prove this using an induction on the number of zones of P' .

For the basis of the induction we consider the smallest zonohedron P' (which is a cube) with three zones. Clearly, in a cube there are four possible c each of which divides the faces of the cube into two sets of faces P'_1 and P'_2 where faces in P'_1 (P'_2) have parallel pairs in P'_2 (P'_1). For each such c there exists a d where exactly the faces of P'_1 (P'_2) are visible and the faces of P'_2 (P'_1) are invisible. Moreover, d is the resultant vector of the outer-normals of those visible faces and is not parallel to any face of the cube.

Let P'' be the zonohedron whose zone cycle c' was expanded in direction d' to obtain P' . Let z' be the zone added to P' due to this expansion (see Figure 5(b)).

Remember that in P' , c divides z' into two chains of faces where faces in one chain have parallel pairs in the other chain. So c contains two zone edges of z' . Let e_1 and e_2 be those two edges. Moreover, since c divides the faces of P' into two sets P'_1 and P'_2 where faces in P'_1 (P'_2) have parallel pairs in P'_2 (P'_1), the faces of P'' (without z') are also divided by the cycle $c'' = c' \setminus \{e_1, e_2\}$ into two subsets of faces where faces in one set have parallel pairs in the other set. Therefore, by induction hypothesis there is a direction d'' which is not parallel to any face of P'' and from which all faces in one side of c'' are visible and all faces in the other side of c'' are invisible. Let the set of visible and invisible faces be P''_1 and P''_2 respectively.

For the remaining proof we will switch our attention to the Gauss map. Let g'' be the great circle whose plane is perpendicular to d'' . Let the two half spheres defined by g'' be h_1 and h_2 . Assume that h_1 (h_2) is visible (invisible) to d'' . So the normal-points of the faces of P''_1 and P''_2 are within h_1 and h_2 respectively.

In P'' , c'' and c' must intersect (Figure 5(a)) possibly sharing some edges (Figure 5(d)). Hence we have two cases.

Case 1: c'' and c' intersect in a pair of vertices.

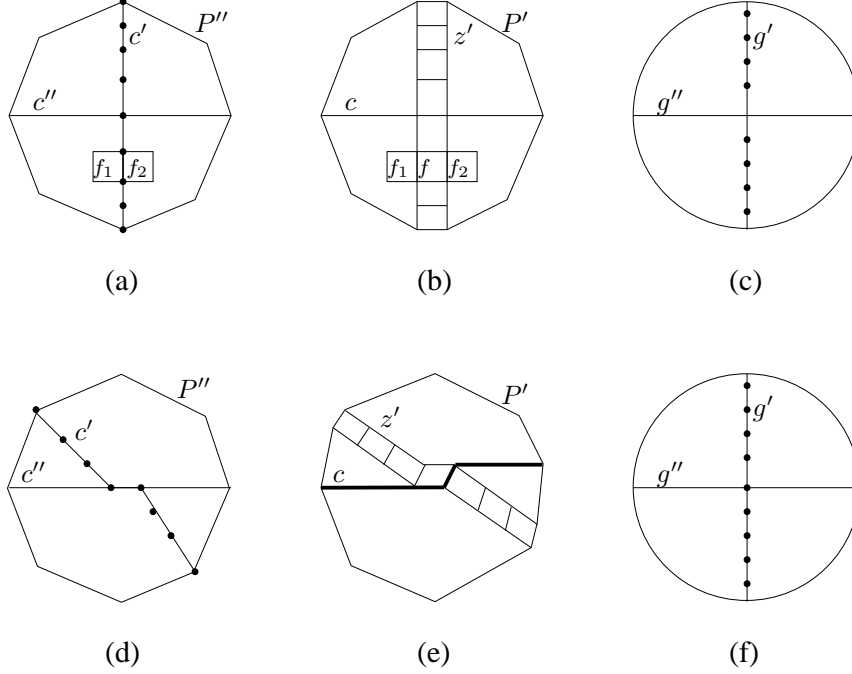


Fig. 5. (a) The two zone cycles c'' and c' in P'' , (b) The corresponding zone z' and cycle c in P' , (c) The great circles g'' and g' representing c'' and c' in the Gauss map (for Case 1), (d) The two cycles c'' and c' sharing two edges e, e' in P'' , (e) The corresponding zone z' and cycle c in P' and (f) The great circles g'' and g' representing c'' and c' in the Gauss map (for Case 2).

For this case we will prove that d'' will work as d . Let f_1, f_2 be two arbitrary adjacent faces of P'' whose common edge e is in c' . Assume that the normal points of f_1 and f_2 are in h_1 (similarly in h_2). After expanding P'' to P' , let the face created from e be f . Since by expansion the normal point of f_1 and f_2 remain unchanged, it suffices to prove that the normal point of f is also in $h_1(h_2)$ (see Figure 5(b)). Since f_1, f, f_2 are three adjacent faces of a zone (other than z') of P' , their normal points must lie on a great circle and the normal-point of f is in the geodesic arc connecting that of f_1 and f_2 . Therefore, the normal-point of f must be within $h_1(h_2)$.

Case 2: c'' and c' share some edges.

We will first prove that c'' and c' share exactly two edges. Let the two great circles of d'' and d' be g'' and g' respectively. Edges/vertices of c'' (c') represent points/arcs of g'' (g') respectively. (For c' simply think its edges/vertices as the zone faces/zone edges of z' and for c'' simply think the edges/vertices of c'' as the zone faces/zone edges of the zone that would be created if c'' were expanded in direction d''). Now, g'' and g' intersect into two antipodal points and their corresponding two edges are only common in c'' and c' . See Figure 5(d,e,f).

After creating z' the two common edges become two parallel faces. Let they be

f and f' . By the argument of Case 1, except f and f' all faces in one side of c are visible and all faces in the other side are invisible from d'' . If f and f' too are not parallel to d'' and are visible/invisible as required, then $d = d''$, and we are done. Otherwise, we can always take d as $d'' + \epsilon$, where ϵ is small enough such that f and f' are no more parallel to d , they become visible/invisible as required, and the visibility/invisibility of all other faces remain the same. Note that there may be one more case: it may be possible that $f(f')$ is supposed to be visible (invisible) but is invisible (visible) from d'' . Then by symmetry of P' we can simply interchange f and f' in P'_1 and P'_2 and thus take $d = d''$.

Now d can be easily found as follows. From the above argument it is clear that there exists a d from which all the faces in one side of c are visible. In fact d is a direction in the intersection of the positive half-spaces (the positive half-space of a face f is the plane of f from which f is visible) of the faces of P_1 adjacent to c . Hence determining d takes $O(h \log h)$ time [1] where h is the number of edges of c . \square

4.3 Running Time

Now we examine the time complexity of the construction as a whole. Finding the zones and the face pairs take linear time. Deletion of zones also takes linear time. Final points of P are calculated by the amount of expansion of all zones of G by Lemma 4.4. Thus the total time required for all expansion is $O(h_1 \log h_1 + h_2 \log h_2 + \dots + h_m \log h_m)$, where m is the number of zones of G and h_i is the number of faces of the new zone at i -th expansion. Since all the faces are quadrilaterals, at i -th step, h_i new faces are created. Hence the sum $h_1 + h_2 + \dots + h_m$ is the total number of faces which is $O(n)$, where n is the number of vertices in G . Moreover, $h_1 = 4$ and from Lemma 3.2, $h_i = h_{i-1} + 2$, which implies that $m = O(\sqrt{n})$.

Theorem 4.5 *The number of zones in a zonohedron is $O(\sqrt{n})$.*

Corollary 1 *The maximum number of faces in a zone is $O(\sqrt{n})$.*

Proof. By Lemma 3.2, every two zones intersects into two parallel faces. So a zone can intersect $O(\sqrt{n})$ other zones in $O(\sqrt{n})$ faces. \square

Therefore, the total running time of the construction algorithm is $O(h_m \log(h_1 \cdot h_2 \cdot \dots \cdot h_m)) = O(\sqrt{n} \log(4 \cdot 6 \cdot 8 \cdot \dots \cdot (4 + 2m))) = O(\sqrt{n} \log(2^m(m+2)!)) = O(\sqrt{n}(m \log 2 + m \log m)) = O(\sqrt{n}(\sqrt{n} + \sqrt{n} \log n)) = O(n \log n)$.

Theorem 4.6 *A zonohedron P from a zonohedral graph G can be constructed in $O(n \log n)$ time, where n is the number of vertices in G .*

5 Recognizing a zonohedral graph

Let G be the given graph. G can be tested for 3-connected planar in linear time [11]. Testing whether all faces of G are even takes linear time. We already discussed that finding zones and face pairs takes linear time. Once the face pairs are determined, we can measure how a zone is divided by its face pairs. For all zones it takes linear time in total.

Theorem 5.1 *Given a graph G , recognizing whether G is zonohedral can be done in linear time.*

Observe that our recognition of a zonohedral graph will also work for recognizing the graph of a generalized zonohedron.

Corollary 2 *The graph of a generalized zonohedron can be recognized in linear time.*

6 Conclusion

An immediate open problem is to characterize graphs of other subclasses of convex polyhedra, in particular graphs of generalized zonohedra. A generalized zonohedron contains faces of length greater than four. The difficulty with characterizing graphs of generalized zonohedra is that after deletion of a zone the cycle c may contain faces. Hence during the construction we have to prove that those faces are in parallel pairs and a great circle exists through them.

Our construction of P starts with a cube. But it will also work if we started with a parallelepiped.

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